

# The Myth of Equidistribution for High-Dimensional Simulation\*

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## Abstract

A pseudo-random number generator (RNG) might be used to generate  $w$ -bit random samples in  $d$  dimensions if the number of state bits is at least  $dw$ . Some RNGs perform better than others and the concept of equidistribution has been introduced in the literature in order to rank different RNGs.

We define what it means for a RNG to be  $(d, w)$ -equidistributed, and then argue that  $(d, w)$ -equidistribution is not necessarily a desirable property.

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## 1 Motivation

*There is no such thing as a random number – there are only methods to produce random numbers, and a strict arithmetic procedure of course is not such a method.*

John von Neumann [11, p. 768]

Suppose we are performing a simulation in  $d$  dimensions. For simplicity let the region of interest be the unit hypercube  $H = [0, 1)^d$ .

For the simulation we may need a sequence  $y_0, y_1, \dots$  of points uniformly and independently distributed in  $H$ . A pseudo-random number generator gives us a sequence  $x_0, x_1, \dots$  of points in  $[0, 1)$ . Thus, it is natural to group these points in blocks of  $d$ , that is

$$y_j = (x_{jd}, x_{jd+1}, \dots, x_{jd+d-1}).$$

If our pseudo-random number generator is good and  $d$  is not too large, we expect the  $y_j$  to behave like uniformly and independently distributed points in  $H$ .

## 2 Pseudo-random vs quasi-random

We are considering applications where the (pseudo-)random number generator should, as far as possible, be indistinguishable from a perfectly random source. In some applications, e.g. Monte Carlo quadrature, it is better to use *quasi-random* numbers which are intended for that application and give an estimate with smaller variance than we could expect with a perfectly random source.

For example, when estimating a contour integral of an analytic function, we might transform the contour to a circle and use equally spaced points on the circle.

However, when simulating Canberra's future climate and water supply, it would not be a good idea to assume that exceptionally dry years were equally spaced!

## 3 Goodness of fit

If we use the  $\chi^2$  test to test the hypothesis that a set of data is a random sample from some distribution, then we typically reject the hypothesis if the  $\chi^2$  statistic is *too large*.

However, we should equally reject the hypothesis if  $\chi^2$  is *too small* (because in this case the fit is *too good*) [9].

## 4 Linear congruential generators

In the “old days” people often followed Lehmer’s suggestion and used linear congruential random number generators of the form

$$z_{n+1} = az_n + b \bmod m.$$

This gives an integer in  $[0, m)$  so needs to be scaled:

$$x_n = z_n/m.$$

Typically  $m$  is a power of two such as  $2^{32}$  or  $2^{64}$ , or a prime close to such a power of two.

Unfortunately, all such linear congruential generators perform badly in high dimensions, as shown in Marsaglia’s famous paper *Random numbers fall mainly in the planes* [7].

## 5 RANDU

Some linear congruential generators perform disastrously. For example, consider the infamous RANDU:

$$z_{n+1} = 65539z_n \bmod 2^{31}$$

(with  $z_0$  odd). These points satisfy

$$z_{n+2} - 6z_{n+1} + 9z_n = 0 \bmod 2^{31}$$

so in dimension  $d = 3$  the resulting points  $y_j$  all lie on a small number of planes, in fact 15 planes separated by distance  $1/\sqrt{1^2 + 6^2 + 9^2} \approx 0.092$

In general, such behaviour is detected by the *spectral test* [6].

Even the best linear congruential generators perform badly because they have period at most  $m$ , so the average distance between points  $y_j$  is of order

$$\frac{1}{m^{1/d}}$$

(so the set of points closest to any one  $y_j$  has volume of order  $1/m$ ).

## 6 Modern generators

Nowadays, linear congruential generators are rarely used in high-dimensional simulations. Instead, generators with much longer periods are used. A popular class is those given by a linear recurrence over  $F_2$ . These take the form

$$u_i = Au_{i-1} \bmod 2$$

$$v_i = Bu_i \bmod 2$$

$$x_i = \sum_{j=1}^w v_{i,j} 2^{-j}$$

where  $u_i$  is an  $n$ -bit state vector,  $v_i$  is a  $w$ -bit output vector which may be regarded as a fixed-point number  $x_i$ , and the linear algebra is performed over the field  $F_2 = \text{GF}(2)$  of two elements  $\{0, 1\}$ . Here  $A$  is an  $n \times n$  matrix and  $B$  is a  $w \times n$  matrix (both over  $F_2$ ). Usually  $A$  is sparse (so the matrix-vector multiplication can be performed quickly) and often  $B$  is a projection.

## 7 The period

Provided the characteristic polynomial of  $A$  is primitive over  $F_2$ , and  $B \neq 0$ , the period of such a generator is  $2^n - 1$ . This can be very large, e.g.  $n = 4096$  for *xorgens* [3] and  $n = 19937$  for the *Mersenne Twister* [8]. For details we refer to L'Ecuyer's papers [5, 12].

## 8 Equidistribution

Various definitions of  $(d, w)$ -equidistribution can be found in the literature. We follow Panneton and L'Ecuyer [12] without attempting to be too general.

Consider  $w$ -bit fixed-point numbers. There are  $2^w$  such numbers in  $[0, 1)$ . Each such number can be regarded as representing a small interval of length  $2^{-w}$ .

Similarly, in  $d$  dimensions, we can consider small hypercubes whose sides have length  $2^{-w}$ . Each small hypercube has volume  $2^{-dw}$  and there are  $2^{dw}$  of them in the unit hypercube  $[0, 1)^d$ . A small hypercube can be specified by a  $d$ -dimensional vector of  $w$ -bit numbers (a total of  $dw$  bits).

## Definition

Consider a random number generator with period  $2^n$ . (A slight change in the definition can be made to accomodate generators with period  $2^n - 1$ .)

If the generator is run for a complete period to generate  $2^n$  pseudo-random points in  $[0, 1]^d$ , we say that the generator is  $(d, w)$ -equidistributed if the same number of points fall in each small hypercube.

The condition  $n \geq dw$  is necessary. The number of points in each small hypercube is  $2^{n-dw}$ .

RANDU (with  $n = 29$ ) is *not*  $(d, w)$ -equidistributed for any  $d \geq 3, w \geq 4$ . However, most good long-period generators *are*  $(d, w)$ -equidistributed for  $dw \ll n$ .

## 9 Figures of merit

The maximum  $w$  for which a generator can be  $(d, w)$ -equidistributed is  $w_d^* = \lfloor n/d \rfloor$ . If a generator is actually  $(d, w)$ -equidistributed for  $w \leq w_d$  then

$$\delta_d = w_d^* - w_d$$

is sometimes called the “resolution gap” [5] and

$$\Delta = \max_{d \leq n} \delta_d$$

is taken as a figure-of-merit (small  $\Delta$  is desirable). However, this only makes sense when comparing generators with the same period. When comparing generators with different periods, it makes more sense to consider

$$W = \sum_{d \leq n} w_d$$

as a figure of merit (a large value is desirable). An upper bound is  $W \leq \sum_d w_d^* \sim n \ln n$ .

## 10 Problems with equidistribution

A test for randomness should (usually) be passed by a perfectly random source.

$(d, w)$ -equidistribution applies only to a periodic sequence: we need to know the period  $N = 2^n$  (or  $N = 2^n - 1$ ). A perfectly random source

is not periodic, but we can get a periodic sequence by taking the first  $N$  elements  $(y_0, y_1, \dots, y_{N-1})$  and then repeating them ( $y_{i+N} = y_i$ ). However, this sequence is unlikely to be  $(d, w)$ -equi-distributed unless  $d$  and  $w$  are very small.

Consider the simplest case  $dw = n$ . There are  $N = 2^n$  small hypercubes and  $N!$  ways in which each of these can be hit by exactly one of  $(y_0, \dots, y_{N-1})$  out of  $N^N$  possibilities. Thus the probability of equidistribution is

$$\frac{N!}{N^N} \sim \frac{\sqrt{2\pi N}}{\exp(N)}.$$

Recall that  $N = 2^n$  is typically very large (for example  $2^{4096}$ ) so  $\exp(N)$  is gigantic.

## Independence of ordering

$(d, w)$ -equidistribution is independent of the ordering of  $y_0, \dots, y_{N-1}$ .

Given a  $(d, w)$ -equidistributed sequence, we can reorder it in any manner and the new sequence will still be  $(d, w)$ -equidistributed.

For example,  $y_j = j \bmod 2^n$  gives a  $(1, n)$ -equidistributed sequence.

## A common argument

It is often argued that, when  $n$  is large, we will not use the full sequence of length  $N = 2^n$ , but just some initial segment of length  $M \ll N$ . If  $M \ll \sqrt{N}$  then the initial segment may behave like the initial segment of a random sequence. However, if this is true, what is the benefit of  $(d, w)$ -equidistribution?

## 11 Why consider equidistribution?

The main argument in favour of considering equidistribution seems to be that, for several popular classes of pseudo-random number generators, we can test if the sequence is  $(d, w)$ -equidistributed without actually generating a complete cycle of length  $N$ .

For generators given by a linear recurrence over  $F_2$ ,  $(d, w)$ -equidistribution is equivalent to a certain matrix over  $F_2$  having full rank. However, the fact that a property is easily checked does not mean that it is relevant. We actually need something weaker (but harder to check).

## 12 Conclusion

When comparing modern long-period pseudo-random number generators,  $(d, w)$ -equidistribution is irrelevant, because it is neither necessary nor sufficient for a good generator.

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